# On Chvátal's conjecture and a conjecture on families of signed sets

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**Abstract**

A family $\mathcal{H}$ of sets is said to be *hereditary* if all subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$; in other words, $\mathcal{H}$ is hereditary if it is a union of *power sets*. A family $\mathcal{A}$ is said to be *intersecting* if no two sets in $\mathcal{A}$ are disjoint. A *star* is a family whose sets contain at least one common element. An outstanding open conjecture due to Chvátal claims that among the largest intersecting sub-families of any finite hereditary family there is a star. We suggest a weighted version that generalises both Chvátal's conjecture and a conjecture (due to the author) on intersecting families of *signed sets*. Also, we prove the new conjecture for weighted hereditary families that have a *dominant element*, hence generalising various results in the literature.

On Chvátal's conjecture

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# 1 Some basic definitions and notation

We shall use small letters such as $x$ to denote elements of a set or non-negative integers or functions, capital letters such as $X$ to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (i.e. sets whose members are sets themselves). Unless otherwise stated, it is to be assumed that sets and families (and sets in families) are *finite*.

For any integer $n \geq 1$, the set $\{1, ..., n\}$ of the first $n$ positive integers is denoted by $[n]$. For a set $X$, the *power set of* $X$ (i.e. the family of all subsets of $X$) is denoted by $2^X$, and the family of all $r$-element subsets of $X$ is denoted by $\binom{X}{r}$. An *$r$-set* is a set of size $r$.

We denote the union of all sets in a family $\mathcal{F}$ by $U(\mathcal{F})$. For any $x \in U(\mathcal{F})$, we denote the family of those sets in $\mathcal{F}$ which contain $x$ by $\mathcal{F}\langle x \rangle$.

A family $\mathcal{H}$ is said to be a *hereditary family* (also called an *ideal* or a *downset*) if all the subsets of any set in $\mathcal{H}$ are in $\mathcal{H}$. Clearly a family is hereditary if and only if it is a union of power sets. A *base* of $\mathcal{H}$ is a set in $\mathcal{H}$ that is not a subset of any other set in $\mathcal{H}$. So a hereditary family is the union of power sets of its bases.

A family $\mathcal{A}$ is said to be *intersecting* if any two sets in $\mathcal{A}$ contain at least one common element. If the sets in a family $\mathcal{A}$ have a common element $x$ (i.e. $\mathcal{A} = \mathcal{A}\langle x \rangle$), then $\mathcal{A}$ is said to be a *star*. So a star is an intersecting family. The simplest example of an intersecting family that is not a star is $\{\{1,2\}, \{1,3\}, \{2,3\}\}$ (i.e. $\binom{[3]}{2}$).

If $U(\mathcal{F})$ contains an element $x$ such that $\mathcal{F}\langle x \rangle$ is a largest intersecting sub-family of $\mathcal{F}$ (i.e. no intersecting sub-family of $\mathcal{F}$ has more sets than $\mathcal{F}\langle x \rangle$), then we say that $\mathcal{F}$ has the *star property at* $x$. We simply say that $\mathcal{F}$ has the *star property* if either $U(\mathcal{F})$ is the empty set $\emptyset$ or $\mathcal{F}$ has the star property at some element of $U(\mathcal{F})$.

For a non-empty set $X$ and $x, y \in X$, let $\lambda_{x,y} \colon 2^X \to 2^X$ be defined by

$$\lambda_{x,y}(A) = \begin{cases} (A \backslash \{y\}) \cup \{x\} & \text{if } y \in A \text{ and } x \notin A; \\ A & \text{otherwise,} \end{cases}$$

and let $\Lambda_{x,y} \colon 2^{2^X} \to 2^{2^X}$ be the *compression operation* defined by

$$\Lambda_{x,y}(\mathcal{A}) = \{\lambda_{x,y}(A) \colon A \in \mathcal{A}, \lambda_{x,y}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \lambda_{x,y}(A) \in \mathcal{A}\}.$$

Note that $|\Lambda_{x,y}(\mathcal{A})| = |\mathcal{A}|$. It is well-known, and easy to check, that $\Lambda_{x,y}(\mathcal{A})$ is intersecting if $\mathcal{A}$ is intersecting; [15] is an excellent survey on the properties and uses of compression (also called *shifting*) operations in extremal set theory.

If $x \in U(\mathcal{F})$ such that $\lambda_{x,y}(F) \in \mathcal{F}$ for any $F \in \mathcal{F}$ and any $y \in U(\mathcal{F})$, then $\mathcal{F}$ is said to be *compressed with respect to* $x$. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be *left-compressed* if $\lambda_{i,j}(F) \in \mathcal{F}$ for any $F \in \mathcal{F}$ and any $i, j \in [n]$ with $i < j$.

# 2 Intersecting sub-families of hereditary families

The following is a famous longstanding open conjecture in extremal set theory due to Chvátal.

**Conjecture 2.1 ([9])** *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ has the star property.*

This conjecture was verified for the case when $\mathcal{H}$ is left-compressed by Chvátal [10] himself. Snevily [24] took this result (together with results in [23, 25]) a significant step forward by verifying Conjecture 2.1 for the case when $\mathcal{H}$ is compressed with respect to an element $x$ of $U(\mathcal{H})$.

**Theorem 2.2 ([24])** *If a hereditary family $\mathcal{H}$ is compressed with respect to an element $x$ of $U(\mathcal{H})$, then $\mathcal{H}$ has the star property at $x$.*

A special case is when the bases of $\mathcal{H}$ contain a common element; this was settled in [23].

Snevily's proof of Theorem 2.2 makes use of the following interesting result of Berge [2].

**Theorem 2.3 ([2])** *If $\mathcal{H}$ is a hereditary family, then $\mathcal{H}$ is a disjoint union of pairs of disjoint sets, together with $\emptyset$ if $|\mathcal{H}|$ is odd.*

This result was also motivated by Conjecture 2.1, and it implies that the size of an intersecting sub-family of a hereditary family $\mathcal{H}$ cannot be greater than $|\mathcal{H}|/2$.

For any integer $s \geq 0$, let $\mathcal{H}^{(s)} = \{H \in \mathcal{H} \colon |H| = s\}$ and $\mathcal{H}^{(\leq s)} = \{H \in \mathcal{H} \colon |H| \leq s\}$. In [5] it is shown that if the size of any base of a hereditary family $\mathcal{H}$ is at least $\frac{3}{2}(r-1)^2(3r-4)+r$, then for any $S \subseteq [r]$, the union $\bigcup_{s \in S} \mathcal{H}^{(s)}$ has the star property, and hence the *level* $\mathcal{H}^{(r)}$ and the hereditary sub-family $\mathcal{H}^{(\leq r)}$ of $\mathcal{H}$ have the star property.

Many other results have been inspired by Conjecture 2.1; see [8, 19]. Interesting variations on this conjecture have been suggested by Snevily; see [26].

## 3 Intersecting families of signed sets

Let $x_1, ..., x_r$ be the distinct elements of an $r$-set $X$, and let $y_1, ..., y_r, k$ be integers satisfying $1 \leq y_i \leq k$ for all $i \in [r]$. We call the $r$-set $\{(x_1, y_1), ..., (x_r, y_r)\}$ a *$k$-signed set on $X$*. For any integer $k \geq 1$, we denote the family of all $k$-signed sets on $X$ by $\mathcal{S}_{X,k}$, that is,

$$\mathcal{S}_{X,k} = \{\{(x_1, y_1), ..., (x_r, y_r)\} \colon y_1, ..., y_r \in [k]\}.$$

We shall set $\mathcal{S}_{\emptyset,k} = \emptyset$. For any family $\mathcal{F}$, we denote the union of all families $\mathcal{S}_{F,k}$ with $F \in \mathcal{F}$ by $\mathcal{S}_{\mathcal{F},k}$, that is,

$$\mathcal{S}_{\mathcal{F},k} = \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

The 'signed sets' terminology was introduced in [4] for a setting that can be re-formulated as $\mathcal{S}_{\binom{[n]}{r},k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced in [6], the theme of which is the following conjecture.

**Conjecture 3.1 ([6])** *For any family $\mathcal{F}$ and any integer $k \geq 2$, $\mathcal{S}_{\mathcal{F},k}$ has the star property.*

Obviously we cannot replace $k \geq 2$ by $k \geq 1$, because if $\mathcal{F}$ does not have the star property (for example, $\mathcal{F}$ is a non-star intersecting family such as $\binom{[3]}{2}$), then neither does $\mathcal{S}_{\mathcal{F},1}$ (since $\mathcal{F}$ and $\mathcal{S}_{\mathcal{F},1}$ have the same structure). The main result in the same paper is that this conjecture is true if $\mathcal{F}$ is compressed with respect to an element $x$ of $U(\mathcal{F})$.

**Theorem 3.2 ([6])** *If a family $\mathcal{F}$ is compressed with respect to an element $x$ of $U(\mathcal{F})$, then $\mathcal{S}_{\mathcal{F},k}$ has the star property at $(x, 1)$ for any $k \geq 2$.*

This generalises a well-known result that was first stated by Meyer [20] and proved in different ways by Deza and Frankl [11], Bollobás and Leader [4], Engel [12] and Erdős et al. [13], and that can be described as saying that the conjecture is true for $\mathcal{F} = \binom{[n]}{r}$. Berge [3] and Livingston [22] had proved this for the special case $\mathcal{F} = \{[n]\}$ (other proofs are found in [16, 21]). In [6] the conjecture is also verified for families $\mathcal{F}$ that are uniform (i.e. their sets are of equal size) and have the star property; Holroyd and Talbot [17] had essentially proved this in a graph-theoretical context. In [7] the conjecture is proved for $k$ sufficiently large, depending only on the size of a largest set in $\mathcal{F}$.

**Theorem 3.3 ([7])** *Let $\alpha_{\mathcal{F}}$ be the size of a largest set in a family $\mathcal{F}$. For any integer $k \geq \max\{1, (\alpha_{\mathcal{F}})^2(\alpha_{\mathcal{F}} - 1)/2\}$, $\mathcal{S}_{\mathcal{F},k}$ has the star property.*

# 4 Intersecting sub-families of weighted hereditary families

Let $\mathbb{R}$ denote the set of real numbers. For any family $\mathcal{F}$ and any function $w\colon \mathcal{F} \to \mathbb{R}$ (which we call a *weight function*), we denote the sum $\sum_{F\in\mathcal{F}} w(F)$ (of *weights* of sets in $\mathcal{F}$) by $w(\mathcal{F})$. If $U(\mathcal{F})$ contains an element $x$ such that $w(\mathcal{A}) \leq w(\mathcal{F}\langle x\rangle)$ for any intersecting sub-family $\mathcal{A}$ of $\mathcal{F}$, then we say that $(\mathcal{F}, w)$ has the *weighted star property at* $x$. We simply say that $(\mathcal{F}, w)$ has the *weighted star property* if either $U(\mathcal{F}) = \emptyset$ or $(\mathcal{F}, w)$ has the weighted star property at some element of $U(\mathcal{F})$.

We suggest a conjecture that relates Conjectures 2.1 and 3.1 in the sense that it provides a common generalisation.

**Conjecture 4.1** *If $\mathcal{H}$ is a hereditary family and $w\colon \mathcal{H} \to \mathbb{R}$ such that $w(H) \geq w(H')$ for any $H, H' \in \mathcal{H}$ with $H \subseteq H'$, then $(\mathcal{H}, w)$ has the weighted star property.*

**Theorem 4.2** *If Conjecture 4.1 is true, then Conjectures 2.1 and 3.1 are true.*

**Proof.** Suppose Conjecture 4.1 is true. Then Conjecture 2.1 follows by taking $w(H) = 1$ for all $H \in \mathcal{H}$, and Conjecture 3.1 follows immediately from the following lemma. $\Box$

**Lemma 4.3** *Let $\mathcal{F}$ be a family, and let $\mathcal{H} = \bigcup_{F\in\mathcal{F}} 2^F$. For any $H \in \mathcal{H}$, let $\mathcal{F}_H = \{F \in \mathcal{F} \colon H \subseteq F\}$. Let $k \geq 2$ be an integer. Let $w\colon \mathcal{H} \to \mathbb{R}$ such that for any $H \in \mathcal{H}$,*

$$w(H) = \left| \bigcup_{F\in\mathcal{F}_H} \{S \in \mathcal{S}_{F,k} \colon S \cap (F \times [1]) = H \times [1]\} \right|.$$

*Then:*
*(i) $\mathcal{H}$ is hereditary;*
*(ii) $w(H) \geq w(H')$ for any $H, H' \in \mathcal{H}$ with $H \subseteq H'$;*
*(iii) if $(\mathcal{H}, w)$ has the weighted star property at an element $x$ of $U(\mathcal{H})$, then $\mathcal{S}_{\mathcal{F},k}$ has the star property at $(x, 1)$.*

This lemma is proved in the next section.

If a family $\mathcal{F}$ is compressed with respect to an element $x$ of $U(\mathcal{F})$ and $w(F) \leq w(\lambda_{x,y}(F))$ for any $F \in \mathcal{F}$ and any $y \in U(\mathcal{F})$, then we say that $x$ is a *dominant element of* $U(\mathcal{F})$ *under* $w$.

The following is our main result, which establishes Conjecture 4.1 for the case when $U(\mathcal{H})$ has a dominant element under $w$.

**Theorem 4.4** *Let $\mathcal{H}$ be a hereditary family, and let $w\colon \mathcal{H} \to \mathbb{R}$ such that $w(H) \geq w(H')$ for any $H, H' \in \mathcal{H}$ with $H \subseteq H'$. If $U(\mathcal{H})$ has a dominant element $x$ under $w$, then $(\mathcal{H}, w)$ has the weighted star property at $x$.*

**Proof.** We use induction on $|U(\mathcal{H})|$. The case $|U(\mathcal{H})| \leq 2$ is trivial, so we assume $|U(\mathcal{H})| > 2$. Suppose $U(\mathcal{H})$ has a dominant element $x$ under $w$. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{H}$. Let $y \in U(\mathcal{H})\backslash\{x\}$, and let $\mathcal{B} = \Lambda_{x,y}(\mathcal{A})$. So $\mathcal{B}$ is intersecting. Since $x$ is a dominant element of $U(\mathcal{H})$ under $w$, we have $\mathcal{B} \subset \mathcal{H}$ and $w(\mathcal{A}) \leq w(\mathcal{B})$.

Let $\mathcal{I} = \mathcal{H}\langle y\rangle$, $\mathcal{I}' = \{I\backslash\{y\}\colon I \in \mathcal{I}\}$ and $\mathcal{J} = \mathcal{H}\backslash\mathcal{H}\langle y\rangle = \{H \in \mathcal{H}\colon y \notin H\}$. Since $\mathcal{H}$ is hereditary, $\mathcal{I}'$ and $\mathcal{J}$ are hereditary, and $\mathcal{I}' \subseteq \mathcal{J}$. Define $v\colon \mathcal{I}' \to \mathbb{R}$ by $v(I) = w(I \cup \{y\})$ $(I \in \mathcal{I}')$; so $v(I) \geq v(I')$ for any $I, I' \in \mathcal{I}'$ with $I \subseteq I'$. Note that $x$ is a dominant element of $U(\mathcal{I}')$ under $v$ and that $x$ is a dominant element of $U(\mathcal{J})$ under $w$.

Let $\mathcal{C} = \{B \in \mathcal{B}\langle y\rangle\colon x \in B,\ B \cap B' = \{y\}$ for some $B' \in \mathcal{B}\langle y\rangle\}$ and $\mathcal{D} = \mathcal{B}\langle y\rangle\backslash\mathcal{C}$. Let $\mathcal{C}' = \{C\backslash\{y\}\colon C \in \mathcal{C}\}$, $\mathcal{D}' = \{D\backslash\{y\}\colon D \in \mathcal{D}\}$ and $\mathcal{E} = \mathcal{B}\backslash\mathcal{B}\langle y\rangle = \{B \in \mathcal{B}\colon y \notin B\}$. So $\mathcal{C}', \mathcal{D}' \subseteq \mathcal{I}'$ and $\mathcal{E} \subseteq \mathcal{J}$. Taking $\mathcal{F} = \mathcal{C}' \cup \mathcal{E}$, we have $\mathcal{F} \subseteq \mathcal{J}$ as $\mathcal{I}' \subseteq \mathcal{J}$.

Suppose $A \cap B = \{y\}$ for some $A, B \in \mathcal{D}$. Then, by definition of $\mathcal{D}$, we have $x \notin A$ and $x \notin B$. Since $\mathcal{B} = \Lambda_{x,y}(\mathcal{A})$, $\lambda_{x,y}(B) \in \mathcal{B}$. But $A \cap \lambda_{x,y}(B) = \emptyset$, which is a contradiction as $\mathcal{B}$ is intersecting. So $(A \cap B)\backslash\{y\} \neq \emptyset$ for any $A, B \in \mathcal{D}$. It follows that $\mathcal{D}'$ is intersecting.

Suppose $A \cap B = \emptyset$ for some $A, B \in \mathcal{F}$. Since $\mathcal{E}$ is an intersecting family (as $\mathcal{E} \subseteq \mathcal{B}$) and each set in $\mathcal{C}'$ contains $x$, one of $A$ and $B$ is in $\mathcal{E}$ and the other is in $\mathcal{C}'$; say $A \in \mathcal{E}$ and $B \in \mathcal{C}'$. But then $A \cap (B \cup \{y\}) = \emptyset$ and $A, B \cup \{y\} \in \mathcal{B}$, which is a contradiction as $\mathcal{B}$ is intersecting. So $\mathcal{F}$ is intersecting.

Since $|U(\mathcal{I}')|$ and $|U(\mathcal{J})|$ are at most $|U(\mathcal{H})\backslash\{y\}| = |U(\mathcal{H})| - 1$, we can now apply the inductive hypothesis to obtain $v(\mathcal{D}') \leq v(\mathcal{I}'\langle x\rangle)$ and $w(\mathcal{F}) \leq w(\mathcal{J}\langle x\rangle)$. Since $v(\mathcal{D}') = w(\mathcal{D})$ and $v(\mathcal{I}'\langle x\rangle) = w(\mathcal{I}\langle x\rangle)$, we have $w(\mathcal{D}) \leq w(\mathcal{I}\langle x\rangle)$.

Suppose $\mathcal{C}' \cap \mathcal{E}$ contains a set $A$. So $A \in \mathcal{B}$. Let $B = A \cup \{y\}$. Then $B \in \mathcal{C}$ and hence $B \cap B' = \{y\}$ for some $B' \in \mathcal{B}$. But then $A \cap B' = \emptyset$, which is a contradiction since $\mathcal{B}$ is intersecting. So $\mathcal{C}' \cap \mathcal{E} = \emptyset$ and hence $|\mathcal{F}| = |\mathcal{C}'| + |\mathcal{E}|$. Therefore $w(\mathcal{F}) = w(\mathcal{C}') + w(\mathcal{E})$.

Bringing all the pieces together and noting that $w(\mathcal{C}) \leq w(\mathcal{C}')$ (by the condition on $w$), we obtain

$$\begin{aligned} w(\mathcal{A}) &\leq w(\mathcal{B}) = w(\mathcal{C}) + w(\mathcal{D}) + w(\mathcal{E}) \leq w(\mathcal{C}') + w(\mathcal{I}\langle x\rangle) + w(\mathcal{E}) \\ &= w(\mathcal{I}\langle x\rangle) + w(\mathcal{F}) \leq w(\mathcal{I}\langle x\rangle) + w(\mathcal{J}\langle x\rangle) = w(\mathcal{H}\langle x\rangle) \end{aligned}$$

as required. □

The argument in the above proof is an alternative for the one used by Snevily [24] for the proof of Theorem 2.2 (and which employs Theorem 2.3); note that Theorem 2.2 follows from Theorem 4.4 by taking $w(H) = 1$ for all $H \in \mathcal{H}$. Theorem 3.2 follows from Theorem 4.4 via Lemma 4.3. Theorem 4.4 also has the following consequence.

**Corollary 4.5 (See [1, 14])** *Let $w\colon 2^{[n]} \to \mathbb{R}$ such that $w(A) \geq w(B)$ for any $A, B \in 2^{[n]}$ with $|A| \leq |B|$. Then $(2^{[n]}, w)$ has the weighted star property at any element of $[n]$.*

**Proof.** Obviously $2^{[n]}$ is hereditary and $w$ obeys the condition in Theorem 4.4. Now let $x \in [n]$. Let $C \in 2^{[n]}$, $y \in [n]$, $D = \lambda_{x,y}(C)$. Since $|D| = |C|$, the condition on $w$ gives us $w(D) \geq w(C)$ and $w(C) \geq w(D)$; hence $w(C) = w(\lambda_{x,y}(C))$. So $x$ is a dominant element of $U(2^{[n]}) = [n]$ under $w$. The result now follows by Theorem 4.4. □

A nice application of this result is given in [18].

# 5 Proof of Lemma 4.3

For an $n$-set $X = \{x_1, ..., x_n\}$ and $(a, b) \in X \times [k]$, let $\delta_{a,b}\colon \mathcal{S}_{2^X,k} \to \mathcal{S}_{2^X,k}$ be defined by

$$\delta_{a,b}(A) = \begin{cases} (A\backslash\{(a,b)\}) \cup \{(a,1)\} & \text{if } (a,b) \in A; \\ A & \text{otherwise,} \end{cases}$$

and let $\Delta_{a,b}\colon 2^{\mathcal{S}_{2^X,k}} \to 2^{\mathcal{S}_{2^X,k}}$ be the compression operation defined by

$$\Delta_{a,b}(\mathcal{A}) = \{\delta_{a,b}(A)\colon A \in \mathcal{A}, \delta_{a,b}(A) \notin \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{a,b}(A) \in \mathcal{A}\}.$$

Note that $|\Delta_{a,b}(\mathcal{A})| = |\mathcal{A}|$ and that, if $\mathcal{F} \subseteq 2^X$ such that $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}$, then $\Delta_{a,b}(\mathcal{A}) \subseteq \mathcal{S}_{\mathcal{F},k}$. As in the case of $\Lambda_{x,y}$, $\Delta_{a,b}(\mathcal{A})$ is intersecting if $\mathcal{A}$ is intersecting; moreover, the following holds (see, for example, [7, Corollary 3.2]).

**Lemma 5.1** *Let $X$ be an $n$-set $\{x_1, ..., x_n\}$, and let $k \geq 2$ be an integer. Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{S}_{2^X,k}$, and let*

$$\mathcal{A}^* = \Delta_{x_n,k} \circ ... \circ \Delta_{x_n,2} \circ ... \circ \Delta_{x_1,k} \circ ... \circ \Delta_{x_1,2}(\mathcal{A}).$$

*Then $A \cap B \cap (X \times [1]) \neq \emptyset$ for any $A, B \in \mathcal{A}^*$.*

**Proof of Lemma 4.3.** (i) Trivial.
(ii) Let $H, H' \in \mathcal{H}$ with $H \subseteq H'$. Then $\mathcal{F}_{H'} \subseteq \mathcal{F}_H$. We have

$$\begin{aligned} w(H') &= \sum_{F \in \mathcal{F}_{H'}} |\{S \in \mathcal{S}_{F,k} \colon S \cap (F \times [1]) = H' \times [1]\}| = \sum_{F \in \mathcal{F}_{H'}} (k-1)^{|F|-|H'|} \\ &\leq \sum_{F \in \mathcal{F}_{H'}} (k-1)^{|F|-|H|} \leq \sum_{F \in \mathcal{F}_H} (k-1)^{|F|-|H|} \\ &= \sum_{F \in \mathcal{F}_H} |\{S \in \mathcal{S}_{F,k} \colon S \cap (F \times [1]) = H \times [1]\}| = w(H). \end{aligned}$$

(iii) Let $\mathcal{A}$ be an intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. Let $\mathcal{A}^*$ be as in Lemma 5.1 with $X = U(\mathcal{F})$. Then $\mathcal{A}^* \subseteq \mathcal{S}_{\mathcal{F},k}$. Let $\mathcal{B} = \{H \in \mathcal{H} \colon A \cap (X \times [1]) = H \times [1]$ for some $A \in \mathcal{A}^*\}$. By Lemma 5.1, $\mathcal{B}$ is an intersecting sub-family of $\mathcal{H}$. Since $\mathcal{A}^* \subseteq \mathcal{S}_{\mathcal{F},k}$, we have $\mathcal{A}^* \subseteq \bigcup_{B \in \mathcal{B}} \bigcup_{F \in \mathcal{F}_B} \{S \in \mathcal{S}_{F,k} \colon S \cap (F \times [1]) = B \times [1]\}$. So $|\mathcal{A}^*| \leq \sum_{B \in \mathcal{B}} w(B) = w(\mathcal{B})$ and hence, since $|\mathcal{A}| = |\mathcal{A}^*|$, $|\mathcal{A}| \leq w(\mathcal{B})$.

Now suppose $(\mathcal{H}, w)$ has the weighted star property at an element $x$ of $U(\mathcal{H})$. Then $w(\mathcal{B}) \leq w(\mathcal{H}\langle x \rangle)$. We have

$$\begin{aligned} w(\mathcal{H}\langle x \rangle) &= \sum_{H \in \mathcal{H}\langle x \rangle} w(H) = \sum_{H \in \mathcal{H}\langle x \rangle} \left| \bigcup_{F \in \mathcal{F}_H} \{S \in \mathcal{S}_{F,k} \colon S \cap (F \times [1]) = H \times [1]\} \right| \\ &= \left| \bigcup_{H \in \mathcal{H}\langle x \rangle} \bigcup_{F \in \mathcal{F}_H} \{S \in \mathcal{S}_{F,k} \colon S \cap (F \times [1]) = H \times [1]\} \right| = |\mathcal{S}_{\mathcal{F},k}\langle (x,1) \rangle| \, . \end{aligned}$$

Thus, since $|\mathcal{A}| \leq w(\mathcal{B}) \leq w(\mathcal{H}\langle x \rangle)$, we have $|\mathcal{A}| \leq |\mathcal{S}_{\mathcal{F},k}\langle (x,1) \rangle|$. Hence the result. □

# References


[1] R. Ahlswede, G.O.H. Katona, Contributions to the geometry of Hamming spaces, Discrete Math. 17 (1977) 1-22.

[2] C. Berge, A theorem related to the Chvátal conjecture, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pp. 35-40. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[3] C. Berge, Nombres de coloration de l'hypergraphe h-parti complet, in: Hypergraph Seminar (Columbus, Ohio 1972), Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, 13-20.

[4] B. Bollobás, I. Leader, An Erdős-Ko-Rado theorem for signed sets, Comput. Math. Appl. 34 (1997) 9-13.

[5] P. Borg, Extremal $t$-intersecting sub-families of hereditary families, J. London Math. Soc. 79 (2009) 167-185.

[6] P. Borg, Intersecting systems of signed sets, Electron. J. Combin. 14 (2007) #R41.

[7] P. Borg, On $t$-intersecting families of signed sets and permutations, Discrete Math. 309 (2009) 3310-3317.

[8] V. Chvátal, http://users.encs.concordia.ca/∼chvatal/conjecture.html.

[9] V. Chvátal, Unsolved Problem No. 7, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, 1974.

[10] V. Chvátal, Intersecting families of edges in hypergraphs having the hereditary property, in: C. Berge, D.K. Ray-Chaudhuri (Eds.), Hypergraph Seminar, Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, 1974, pp. 61-66.

[11] M. Deza, P. Frankl, The Erdős-Ko-Rado theorem - 22 years later, SIAM J. Algebraic Discrete Methods 4 (1983) 419-431.

[12] K. Engel, An Erdős-Ko-Rado theorem for the subcubes of a cube, Combinatorica 4 (1984) 133-140.

[13] P.L. Erdős, U. Faigle, W. Kern, A group-theoretic setting for some intersecting Sperner families, Combin. Probab. Comput. 1 (1992) 323-334.

[14] P.C. Fishburn, P. Frankl, D. Freed, J.C. Lagarias, A.M. Odlyzko, Probabilities for intersecting systems and random subsets of finite sets, SIAM J. Algebr. Discrete Methods 7 (1986) 73-79.

[15] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), Combinatorial Surveys, Cambridge Univ. Press, London/New York, 1987, pp. 81-110.

[16] H.-D.O.F. Gronau, More on the Erdős-Ko-Rado theorem for integer sequences, J. Combin. Theory Ser. A 35 (1983) 279-288.

[17] F.C. Holroyd and J. Talbot, Graphs with the Erdős-Ko-Rado property, Discrete Math. 293 (2005) 165-176.

[18] C. Y. Ku, D. Renshaw, Erdős-Ko-Rado theorems for permutations and set partitions, J. Combin. Theory Ser. A 115 (2008) 1008-1020.

[19] D. Miklós, Some results related to a conjecture of Chvátal, Ph.D. Dissertation, Ohio State University, 1986.

[20] J.-C. Meyer, Quelques problèmes concernant les cliques des hypergraphes $k$-complets et $q$-parti $h$-complets, in: Hypergraph Seminar (Columbus, Ohio 1972), Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, 127-139.

[21] A. Moon, An analogue of the Erdős-Ko-Rado theorem for the Hamming schemes $H(n,q)$, J. Combin. Theory Ser. A 32 (1982) 386-390.

[22] M.L. Livingston, An ordered version of the Erdős-Ko-Rado Theorem, J. Combin. Theory Ser. A 26 (1979), 162-165.

[23] J. Schönheim, Hereditary systems and Chvátal's conjecture, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pp. 537-539. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[24] H.S. Snevily, A new result on Chvátal's conjecture, J. Combin. Theory Ser. A 61 (1992) 137-141.

[25] D.L. Wang and P. Wang, Some results about the Chvátal conjecture, Discrete Math. 24 (1978) 95-101.

[26] D.B. West, http://www.math.uiuc.edu/∼west/regs/chvatal.html.